Fylladio 2

Exercise 2

The volume $V(a,b,c)=abc$
The area $E(a,b,c)=2ab+2bc+2ac$
The length of the interior diagonal $D=\sqrt{a^2+b^2+c^2}$.
We have

$$\frac{dV}{dt}=\frac{\partial V}{\partial a}\frac{da}{dt}+\frac{\partial V}{\partial b}\frac{db}{dt}+\frac{\partial V}{\partial c}\frac{dc}{dt}=bc\frac{da}{dt}+ac\frac{db}{dt}+ab\frac{dc}{dt}$$
$$\frac{dE}{dt}=\frac{\partial E}{\partial a}\frac{da}{dt}+\frac{\partial E}{\partial b}\frac{db}{dt}+\frac{\partial E}{\partial c}\frac{dc}{dt}=2(b+c)\frac{da}{dt}+2(a+c)\frac{db}{dt}+2(b+a)\frac{dc}{dt}$$
$$\frac{dD}{dt}=\frac{\partial D}{\partial a}\frac{da}{dt}+\frac{\partial D}{\partial b}\frac{db}{dt}+\frac{\partial D}{\partial c}\frac{dc}{dt}=\frac{1}{2}\frac{2a}{\sqrt{a^2+b^2+c^2}}\frac{da}{dt}+\frac{1}{2}\frac{2b}{\sqrt{a^2+b^2+c^2}}\frac{db}{dt}+\frac{1}{2}\frac{2c}{\sqrt{a^2+b^2+c^2}}\frac{dc}{dt}$$

In the previous formulas, we set

$$a=1, b=2, c=3$$
$$\frac{da}{dt}=\frac{db}{dt}=1, \frac{dc}{dt}=-3$$

and we have the answer.

Exercise 3

We have (see Figure E3)

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial w}\frac{\partial w}{\partial x}=\frac{\partial f}{\partial u}-\frac{\partial f}{\partial w}$$

Similarly

$$\frac{\partial f}{\partial y}=-\frac{\partial f}{\partial u}+\frac{\partial f}{\partial v}$$
$$\frac{\partial f}{\partial z}=-\frac{\partial f}{\partial v}+\frac{\partial f}{\partial w}$$

Adding the previous relations we have that

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}=0$$

Exercise 4

We have (see Figure E4)

$$\frac{\partial w}{\partial r}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial r}=f_x\frac{\partial x}{\partial r}+f_y\frac{\partial y}{\partial r}=f_x\cos\theta+f_y\sin\theta$$
$$\frac{\partial w}{\partial \theta}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}=f_x\frac{\partial x}{\partial \theta}+f_y\frac{\partial y}{\partial \theta}=-f_x r\sin\theta+f_y r\cos\theta$$

By replacing the relations above in the sum

$$(\frac{\partial w}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial w}{\partial \theta})^2$$

we obtain our result.

Exercise 5

We set $s(x,y) = x - y$ and we have (see Figure E5)

$$\frac{\partial z}{\partial x} = \frac{dz}{ds}\frac{ds}{dx} = \frac{dz}{ds}$$
$$\frac{\partial z}{\partial x} = \frac{dz}{ds}\frac{ds}{dy} = -\frac{dz}{ds}.$$

Therefore

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{dz}{ds} - \frac{dz}{ds} = 0.$$

Exercise 7

We have

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} = (8x - 4y)(-\sin t) + (8y - 4x)\cos t =$$
$$= (8\cos t - 4\sin t)(-\sin t) + (8\sin t - 4\cos t)\cos t =$$
$$= 4\sin^2 t - 4\cos^2 t.$$

Therefore

$$\frac{dT}{dt} = 0 \Leftrightarrow \cos^2 t = \sin^2 t \Leftrightarrow t = \frac{\pi}{4}, \frac{\pi}{2} + \frac{\pi}{4}, \pi + \frac{\pi}{4}, \frac{3\pi}{2} + \frac{\pi}{4}$$

At these points the function T takes its maximal/minimal values.
On the other hand,

$$\frac{d^2T}{dt^2} = 16\sin t\cos t$$

Therefore

$$\frac{d^2T}{dt^2}(\frac{\pi}{4}) > 0, \frac{d^2T}{dt^2}(\frac{3\pi}{4}) < 0, \frac{d^2T}{dt^2}(\frac{5\pi}{4}) > 0, \frac{d^2T}{dt^2}(\frac{7\pi}{4}) < 0$$

Therefore, at $t = \frac{\pi}{4}, \frac{5\pi}{4}$ we have the coldest points while at $t = \frac{5\pi}{4}, \frac{7\pi}{4}$ we have the warmest points.