Exercise 1

Let $f(x,y,z)=\frac{x}{y}+\frac{y}{z}$. The gradient of f is

$$\nabla f(x,y,z)=(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z})=(\frac{1}{y},-\frac{x}{y^2}+\frac{1}{z},-\frac{y}{z^2})\Longrightarrow$$
$$\nabla f(4,2,1)=(\frac{1}{2},0,-2).$$

- Therefore, the directional derivative $D_v f(p)$ takes its maximum value along the direction $\nabla f(4,2,1)=(\frac{1}{2},0,-2)$ (see theory).
- $Df_v(p)$ takes its minimum value along the direction $-\nabla f(4,2,1)=(-\frac{1}{2},0,2)$.
- $Df_v(p)=0$ along a each vector perpendicular to $\nabla f(4,2,1)=(\frac{1}{2},0,-2)$. For example such a vector is $v=(2,0,\frac{1}{2})$ since $\nabla f(4,2,1)\bullet v=(-\frac{1}{2})2+2(\frac{1}{2})=0$.

Furthermore,

$$D_v f(p)=\nabla f(p)\bullet\frac{v}{|v|}.$$

In this folmula, we set $\nabla f(p)=\nabla f(4,2,1)=(\frac{1}{2},0,-2)$, $v=(2,1,3)$, $|v|=\sqrt{2^2+1^1+3^2}=\sqrt{14}$, we calculate $D_v f(p)$.

Exercise 2.

$AB=B-A=(2,1)$. Therefore $\frac{AB}{|AB|}=(2/\sqrt{5},1/\sqrt{5})$, since $|AB|=\sqrt{2^2+1^2}=\sqrt{5}$.

$AC=C-A=(0,4)$. Therefore $\frac{AC}{|AC|}=(0,1)$, since $|AC|=4$.

$AD=D-A=(5,12)$. Therefore $\frac{AD}{|AD|}=(5/13,12/13)$, since $|AD|=13$.

Let $\nabla f(A)=(a,b)$. We have,

$$Df_{AB}(A)=\nabla f(A)\bullet\frac{AB}{|AB|}=(a,b)\bullet(2/\sqrt{5},1/\sqrt{5})=a\frac{2}{\sqrt{5}}+b\frac{1}{\sqrt{5}}=3$$
$$Df_{AC}(A)=\nabla f(A)\bullet\frac{AC}{|AC|}=(a,b)\bullet(0,1)=b=26$$

From these equations we may find a and b.

Also,

$$Df_{AD}(A)=\nabla f(A)\bullet\frac{AD}{|AD|}=(a,b)\bullet(5/13,12/13)=a\frac{5}{13}+b\frac{12}{13}$$

Setting in the last equation the values of a and b that we have found before, we may calculate $Df_{AD}(A)$.

Exercise 3

The equation $x^2+2y^2+3z^2=1$ defines a surface, say S. Let $F(x,y,z)=x^2+2y^2+3z^2$. Then the surface S is defined by the equation $F(x,y,z)=1$.

Let $p = (x, y, z)$ be a point of the surface S. If the tangent plane to S at the point p is parallel to the plane $3x - y + 3z = 1$, then the normal vector of S at p is **parallel** to the normal vector of the plane.

Therefore,

$$\nabla F(x, y, z) = r(3, -1, 3)$$

Therefore,

$$\begin{aligned}(\frac{\partial F}{\partial x}(p), \frac{\partial F}{\partial y}(p), \frac{\partial F}{\partial z}(p)) = r(3, -1, 3) \Longrightarrow \\ (2x, 4y, 6z) = r(3, -1, 3) \Longrightarrow \\ x = \frac{3r}{2}, y = \frac{-r}{4}, z = \frac{r}{2}.\end{aligned}$$

We replace the last values of x, y, z at the equation $x^2 + 2y^2 + 3z^2 = 1$ and we may find r and thus the point $p = (x, y, z)$ where the tangent plane is parallel to $3x - y + 3z = 1$.

Note: Actually you find two values for r and so two points p, p' which have the required property.

Exercise 4

We consider the surface S defined as the graph of the function $z = f(x, y) = \ln(x^2 + y^2)$.

We have $z = \ln(x^2 + y^2) \iff \ln(x^2 + y^2) - z = 0$. Therefore, if we set $F(x, y, z) = \ln(x^2 + y^2) - z$, the surface S is defined as $\{(x, y, z) : F(x, y, z) = \ln(x^2 + y^2) - z = 0\}$.

Now, if $p = (1, 0, 0)$, from our theory we have that

- The tangent plane at p has the equation

$$\frac{\partial F}{\partial x}(p)(x - 1) + \frac{\partial F}{\partial y}(p)(y - 0) + \frac{\partial F}{\partial z}(p)(z - 0) = 0.$$

- The normal line at p has the equation

$$\begin{aligned}x &= 1 + t\frac{\partial F}{\partial x}(p) \\ y &= 0 + t\frac{\partial F}{\partial y}(p) \\ z &= 0 + t\frac{\partial F}{\partial z}(p)\end{aligned}$$

  Therefore you calculate $\frac{\partial F}{\partial x}(p)$, $\frac{\partial F}{\partial y}(p)$, $\frac{\partial F}{\partial z}(p)$, you replace in the previous formulas and you find explicitly the required equations.

Exercise 5

From our theory we know that $Df_v(p)$ takes its maximal value if the vector v is parallel to $\nabla f(p)$.

Therefore, $\nabla f(p) = r(1,1,1)$.

Also, we know that this maximal value is $|\nabla f(p)|$ and on the other hand, it is given in the exercise that this value is $2\sqrt{3}$. Therefore,

$$|\nabla f(p)| = \sqrt{r^2 + r^2 + r^2} = r\sqrt{3} = 2\sqrt{3}$$

Therefore, $r = 2$ and hence $\nabla f(p) = (2,2,2)$.

Furthermore, $D_v f(p)$ is easily calculated since

$$D_v f(p) = \nabla f(p) \bullet \frac{v}{|v|}$$

and $v = (1,1,0)$.

Exercise 6

You find the equation of the normal line to the surface $xy + z = 2$ at $p = (1,1,1)$ and you verify that the point $(0,0,0)$ belongs to this line.

Exercise 7

Let $p = (1,-1,1)$. All the values of $D_v T(p)$ belong to the interval $[-|\nabla T(p)|, |\nabla T(p)|]$.

$\nabla T(x,y,z) = (\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}) = (2y, 2x-z, -y) \Longrightarrow \nabla T(1,-1,1) = (-2,1,1) \Longrightarrow |\nabla T(p)| = \sqrt{6}$.

Since $-3 \notin [-\sqrt{6}, \sqrt{6}]$ the answer is that such a direction u does not exist.